Wiggler-field effects on the space-charge waves of a Raman free-electron laser

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An analysis of the propagation of a space-charge wave through the wiggler and axial magnetic fields of a free-electron laser is presented. The relativistic electron beam is contained within and only partially fills a cylindrical metallic waveguide. A theory is developed using lab-frame Maxwell and fluid equations in a form which is equivalent to the electrostatic approximation in the beam frame. The computational method of determining the dispersion relation is described and some numerical results are presented which illustrate effects arising from the wiggler and the partially filled waveguide. [S1063-651X(99)02208-4]

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I. INTRODUCTION

The relativistic electron beam in a free-electron laser (FEL) passes through a wiggler consisting of a static magnetic field which is periodic along the beam axis. When the FEL is operating in the collective (Raman) regime, the operative mechanism is stimulated Raman scattering. It may be described as the parametric decay of the wiggler field (which is propagating in the beam frame) into a forward scattered space-charge wave and a backscattered electromagnetic wave. A theory of space-charge wave propagation through a wiggler in the presence of an axial magnetic field has been developed by Freund and Sprangle [1] based on the assumption that the beam cross section is infinite. A numerical study by Freund and Antonsen [2] indicates that the combined wiggler and axial fields can have large effects on these waves. Mehdian, Willett, and Aktas [3] have made a recent study based on the infinite-cross-section approximation which shows that the combined wiggler and axial magnetic fields double the number of space-charge modes and modify the electromagnetic modes significantly. The effects of a waveguide boundary on space-charge waves in a wiggler have also been studied recently by Willett and co-workers [4,5] assuming that the guide is completely filled by the electron beam.

The investigation reported herein is an extension of the work of Willett *et al.* [5] to a partially filled waveguide. Propagation of a space-charge wave through a static, spatially periodic magnetic wiggler field and a uniform, static axial magnetic field is analyzed. In Sec. II, the basic differential equations for the fluid and electromagnetic field variables within the electron beam are introduced along with the assumed solutions. In Sec. III, the procedure leading to a system of eleven linear homogeneous algebraic equations in eleven unknown amplitudes is described. The condition for a nontrivial solution of these equations is then invoked to obtain the dispersion relation containing an undetermined radial wave number. In Sec. IV, solutions of Maxwell's equations for the electromagnetic field components in the vacuum region of the waveguide are presented. The boundary conditions are then employed to obtain an auxiliary equation containing the radial wave number required to complete the analysis. In Sec. V, the computational method and the results of a numerical study of the effects of the ratio of the beam radius to waveguide radius, wiggler magnetic field, and axial magnetic field on space-charge waves are described. In Sec. VI, the method of analysis is discussed and some conclusions are presented.

II. SYSTEM OF EQUATIONS AND MODEL

A solid relativistic electron beam with radius a is coaxial with and partially fills a cylindrical metallic waveguide with radius R. The space inside the guide is subject to a static, helical wiggler magnetic field (which is spatially periodic along the guide axis) and a uniform, static axial magnetic field. The wave modes under consideration are electrostatic (potential) waves in the rest frame of electrons, for which their magnetic field and the curl of their electric field are both zero. Analysis will be carried out in the laboratory frame using cylindrical (r,θ,z) coordinates and CGS Gaussian units.

The electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, electron density n, and electron fluid velocity $\mathbf{v}$ within the beam will each be written as an unperturbed part (with subscript zero) plus a small perturbation:

$$\mathbf{E}=\mathbf{E}_0+\delta\mathbf{E}, \tag{1}$$

$$\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}, \tag{2}$$

$$n=n_0+\delta n, \tag{3}$$

$$\mathbf{v}=\mathbf{v}_0+\delta\mathbf{v}. \tag{4}$$

It will be assumed that in the unperturbed state the electron density n_0 is uniform and time independent, the static electric and magnetic fields arising from the unperturbed beam may be neglected, and that the wiggler magnetic field

*Electronic address: behrouz@theory.ipm.ac.ir may be represented by an idealized, one-dimensional approximation. The unperturbed state will be characterized by the following equations:

$$\mathbf{E}_0=\mathbf{0}, \tag{5}$$

$$\mathbf{B}_0=\hat{\mathbf{r}}B_w\cos\Theta+\hat{\boldsymbol{\theta}}B_w\sin\Theta+\hat{\mathbf{z}}B_{0z}, \tag{6}$$

$$\mathbf{v}_0=\hat{\mathbf{r}}v_w\cos\Theta+\hat{\boldsymbol{\theta}}v_w\sin\Theta+\hat{\mathbf{z}}v_\parallel, \tag{7}$$

$$\Theta=k_wz-\theta. \tag{8}$$

Here B_w is the constant magnitude of the wiggler magnetic field, B_{0z} is the constant axial magnetic field, v_w is the constant magnitude of the electron velocity due to the wiggler field, $v_\parallel$ is the constant axial velocity component, and k_w is the constant wiggler wave number. These quantities are related by

$$v_w=\Omega_wv_\parallel/(\Omega_0-k_wv_\parallel), \tag{9}$$

$$\Omega_w=eB_w/(\gamma_0m_0c), \tag{10}$$

$$\Omega_0=eB_{0z}/(\gamma_0m_0c), \tag{11}$$

$$\gamma_0=[1-(v_w^2+v_\parallel^2)c^{-2}]^{-1/2}, \tag{12}$$

where Ω_w and Ω_0 are the relativistic cyclotron frequencies associated with the wiggler and axial magnetic fields, respectively, γ_0 is the unperturbed (constant) Lorentz factor, $e=|e|$ is the magnitude of the electron charge, m_0 is the electron rest mass, and c is the speed of light.

The small perturbations for the density and velocity of the electrons are described by the continuity equation and the relativistic cold-fluid momentum equation

$$\frac{\partial\delta n}{\partial t}+n_0\boldsymbol{\nabla}\cdot\delta\mathbf{v}+(\boldsymbol{\nabla}\delta n)\cdot\mathbf{v}_0=0, \tag{13}$$

$$\begin{aligned}\frac{\partial\delta\mathbf{v}}{\partial t}&+\mathbf{v}_0\cdot\boldsymbol{\nabla}\delta\mathbf{v}+\delta\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}_0\\&=-e(\gamma_0m_0)^{-1}[\delta\mathbf{E}-c^{-2}\mathbf{v}_0\mathbf{v}_0\cdot\delta\mathbf{E}\\&\quad+c^{-1}\delta\mathbf{v}\times\mathbf{B}_0+c^{-1}\mathbf{v}_0\times\delta\mathbf{B}-\gamma_0^2c^{-3}(\mathbf{v}_0\times\mathbf{B}_0)\mathbf{v}_0\cdot\delta\mathbf{v}].\end{aligned} \tag{14}$$

Here $\delta\mathbf{E}$ and $\delta\mathbf{B}$ are the field perturbations of the space-charge wave in the lab frame with the curl of the electric field perturbation as well as the magnetic field perturbation in the beam frame being approximately equal to zero ($\delta\mathbf{B}_B=\mathbf{0}$). Under this electrostatic approximation the Ampère-Maxwell equation is not satisfied exactly and should not be used. Recently, Willett *et al.* [5] have shown that when the electrostatic approximation is used in the beam frame, Gauss's law and $\delta\mathbf{B}_B=\mathbf{0}$ transform to the lab frame as

$$\boldsymbol{\nabla}\cdot\delta\mathbf{E}-\frac{1}{c}\mathbf{v}_\parallel\cdot\left[\boldsymbol{\nabla}\times\delta\mathbf{B}-\left(\frac{4\pi}{c}\delta\mathbf{J}+\frac{1}{c}\frac{\partial\delta\mathbf{E}}{\partial t}\right)\right]=4\pi\delta\rho, \tag{15}$$

$$\gamma_\parallel\left(\delta\mathbf{B}-\frac{1}{c}\mathbf{v}_\parallel\times\delta\mathbf{E}\right)-\frac{\gamma_\parallel^2}{(\gamma_\parallel+1)c^2}\mathbf{v}_\parallel(\mathbf{v}_\parallel\cdot\delta\mathbf{B})=\mathbf{0}, \tag{16}$$

respectively, where $\delta\rho=-e\delta n$ and $\delta\mathbf{J}=-e(n_0\delta\mathbf{v}+\mathbf{v}_0\delta n)$ are the charge and current density perturbations, respectively, and $\gamma_\parallel=(1-v_\parallel^2c^{-2})^{-1/2}$ is the Lorentz factor for the reference-frame transformation. Faraday's law

$$\boldsymbol{\nabla}\times\delta\mathbf{E}=-\frac{1}{c}\frac{\partial\delta\mathbf{B}}{\partial t} \tag{17}$$

must also be employed in the lab-frame analysis.

A solution for a small amplitude wave with radial wave number κ, axial wave number k, and angular frequency ω propagating in the positive z direction will be assumed to be of the form

$$\delta\mathbf{E}=\hat{\mathbf{r}}\delta E_r+\hat{\mathbf{z}}\delta E_z, \tag{18}$$

$$\delta E_r=\delta E_{r0}J_1(\kappa r)\exp[i(kz-\omega t)], \tag{19}$$

$$\delta E_z=\delta E_{z0}J_0(\kappa r)\exp[i(kz-\omega t)], \tag{20}$$

$$\delta\mathbf{B}=\hat{\boldsymbol{\theta}}\delta B_\theta, \tag{21}$$

$$\delta B_\theta=\delta B_{\theta0}J_1(\kappa r)\exp[i(kz-\omega t)], \tag{22}$$

$$\delta n=\delta n_0J_0(\kappa r)\exp[i(kz-\omega t)], \tag{23}$$

$$\delta\mathbf{v}=\hat{\mathbf{r}}\delta v_r+\hat{\boldsymbol{\theta}}\delta v_\theta+\hat{\mathbf{z}}\delta v_z, \tag{24}$$

$$\begin{aligned}\delta v_r=&\{\delta v_{r0}J_1(\kappa r)+(\delta v_{r1}\cos\Theta+\delta v_{r2}\sin\Theta)J_0(\kappa r)\}\\&\times\exp[i(kz-\omega t)],\end{aligned} \tag{25}$$

$$\begin{aligned}\delta v_\theta=&\{\delta v_{\theta0}J_1(\kappa r)+(\delta v_{\theta1}\cos\Theta+\delta v_{\theta2}\sin\Theta)J_0(\kappa r)\}\\&\times\exp[i(kz-\omega t)],\end{aligned} \tag{26}$$

$$\delta v_z=\delta v_{z0}J_0(\kappa r)\exp[i(kz-\omega t)], \tag{27}$$

where J_0 and J_1 are Bessel functions of the first kind of order 0 and 1, respectively. The amplitudes of the wave under consideration are functions of Θ and the radial coordinate r. They may be considered to be represented by a Fourier series in Θ with only the dominant terms retained. The radial dependence of δE_z, δn, and δv_z may be considered to be represented by a Fourier-Bessel series with only one dominant term retained. The required radial dependences of δE_r, δB_θ, δv_r, and δv_θ are then as shown above. Equations (18)–(27) reduce separately to the corresponding solutions in the limits of infinite beam radius [1] and zero wiggler amplitude [6].

A physical explanation for the above solutions may also be presented. Since the wiggler motion of electrons in the unperturbed state is characterized by the terms proportional to $\cos\Theta$ (first order) in the $\hat{\mathbf{r}}$ direction, and $\sin\Theta$ (second order) in the $\hat{\theta}$ direction, with no Θ dependency (zero order) in the $\hat{\mathbf{z}}$ direction, therefore, it is natural to assume the transverse components of the velocity perturbation δv_r and δv_θ

to be affected by the wiggler in a similar manner through the linear combination of $\cos\Theta$ and $\sin\Theta$. The axial velocity δv_z, similar to the z component of $\mathbf{v}_0$, has no Θ dependence. The electric and magnetic field components of the axially symmetric space-charge wave have no Θ dependence; therefore, they should be affected by the wiggler only through their amplitudes and the dispersion relation of the wave.

III. DISPERSION RELATION

In order to derive the dispersion relation for the waves which are electrostatic in the beam frame and have axially symmetric wave field components, the assumed solutions Eqs. (18)–(27) are substituted into Eqs. (13)–(17). Again the Θ dependences will be expanded by the Fourier series. In the radial and azimuthal components of the momentum equation (14) terms up to the second order will be retained. The axial component of the momentum equation (14) and the continuity equation (13) will be written only to zeroth order since δn and δv_z have no Θ dependence. The field equations (15)–(17) are in zeroth order for the axially symmetric electric and magnetic fields.

The above procedure leads to the following eleven linear homogeneous algebraic equations for the eleven unknown amplitudes:

$$k\gamma_\parallel^2\left(\delta E_{r0}-\frac{v_\parallel}{c}\delta B_{\theta 0}\right)+i\gamma_\parallel^2\bar{k}\delta E_{z0}+4\pi e\,\delta n_0 -4\pi e\gamma_\parallel^2 n_0\frac{v_\parallel}{c^2}\delta v_{z0}=0, \tag{28}$$

$$\delta B_{\theta 0}-\frac{v_\parallel}{c}\delta E_{r0}=0, \tag{29}$$

$$ik\,\delta E_{r0}+k\,\delta E_{z0}=i\frac{\omega}{c}\delta B_{\theta 0}, \tag{30}$$

$$-i\bar{\omega}\delta n_0+n_0\kappa\,\delta v_{r0}+ikn_0\,\delta v_{z0}=0, \tag{31}$$

$$-i\bar{\omega}\delta v_{z0}+\frac{e}{m_0\gamma_0\gamma_\parallel^2}\delta E_{z0}-(1/2)\Omega_w\delta v_{\theta 1}+(1/2)\Omega_w\delta v_{r2}=0, \tag{32}$$

$$-i\bar{\omega}\delta v_{r0}+(1/2)v_w\left(\frac{\alpha_3}{\alpha_1}-\kappa\right)\delta v_{r1}-(1/2)v_w\frac{\alpha_3}{\alpha_1}\delta v_{\theta 2} +\frac{i\kappa[1-(1/2)\gamma_\parallel^2 v_w^2 c^{-2}]}{\gamma_0\gamma_\parallel^2\bar{k}}\frac{e}{m_0}\delta E_{z0}+(\Omega_0+\eta/2)\delta v_{\theta 0}=0, \tag{33}$$

$$-i\bar{\omega}\delta v_{\theta 0}+(1/2)v_w\left(\frac{\alpha_3}{\alpha_1}-\kappa\right)\delta v_{\theta 1}+(1/2)v_w\frac{\alpha_3}{\alpha_1}\delta v_{r2} -(\Omega_0+\eta/2)\delta v_{r0}=0, \tag{34}$$

$$-i\bar{\omega}\delta v_{r1}+v_w(\kappa-\alpha_2)\delta v_{r0}+(k_w v_\parallel+\eta/4)\delta v_{r2} -\frac{v_w v_\parallel}{\gamma_0 c^2}\frac{e}{m_0}\delta E_{z0}+(\Omega_0+\eta/4)\delta v_{\theta 1}=0, \tag{35}$$

$$-i\bar{\omega}\delta v_{r2}-v_w\alpha_2\delta v_{\theta 0}-(k_w v_\parallel-\eta/4)\delta v_{r1}-(k_w v_w+\Omega_w -\eta v_\parallel/v_w)\delta v_{z0}+(\Omega_0+3\eta/4)\delta v_{\theta 2}=0, \tag{36}$$

$$-i\bar{\omega}\delta v_{\theta 1}+v_w(\kappa-\alpha_2)\delta v_{\theta 0}+(k_w v_\parallel-\eta/4)\delta v_{\theta 2} -(\Omega_0+3\eta/4)\delta v_{r1}+(k_w v_w+\Omega_w-\eta v_\parallel/v_w)\delta v_{z0} =0, \tag{37}$$

$$-i\bar{\omega}\delta v_{\theta 2}+v_w\alpha_2\delta v_{r0}-(k_w v_\parallel+\eta/4)\delta v_{\theta 1}-\frac{v_w v_\parallel}{\gamma_0 c^2}\frac{e}{m_0}\delta E_{z0} -(\Omega_0+\eta/4)\delta v_{r2}=0. \tag{38}$$

Here

$$\bar{\omega}=\omega-kv_\parallel, \tag{39}$$

$$\bar{k}=k-\omega v_\parallel c^{-2}, \tag{40}$$

$$\eta=-k_w v_\parallel v_w^2\gamma_0^2 c^{-2}, \tag{41}$$

and the radial dependencies are accounted for by the overlap integrals defined as follows:

$$\alpha_1=\frac{\int_0^a rJ_0(\kappa r)J_1(\kappa r)dr}{\int_0^a r[J_0(\kappa r)]^2\,dr}, \tag{42}$$

$$\alpha_2=\frac{\int_0^a J_0(\kappa r)J_1(\kappa r)dr}{\int_0^a r[J_0(\kappa r)]^2\,dr}, \tag{43}$$

$$\alpha_3=\frac{\int_0^a [J_0(\kappa r)]^2 dr}{\int_0^a r[J_0(\kappa r)]^2\,dr}. \tag{44}$$

The dispersion relation is obtained using the necessary and sufficient condition for a nontrivial solution of Eqs. (28)–(38) which after some extensive algebraic manipulation may be written in the following form:

$$\frac{\kappa^2}{\gamma_\parallel^2\bar{k}^2\rho^2}+\frac{(b^2\Omega_0^2-\bar{\omega}^2)(\bar{\omega}^2-\omega_b^2\Phi\gamma_0^{-1}\gamma_\parallel^{-2})}{\bar{\omega}^2(b^2\Omega_0^2+\omega_b^2\Psi\gamma_0^{-1}\gamma_\parallel^{-2}-\bar{\omega}^2)}=0, \tag{45}$$

where

$$\omega_b=(4\pi e^2 n_0/m_0)^{1/2}, \tag{46}$$

$$\rho=\{1+(\Omega_w\Omega_0 v_w v_\parallel^{-1}-\delta_1)[(\Omega_0-k_w v_\parallel)^2-\bar{\omega}^2]^{-1}\}^{1/2}, \tag{47}$$

$$b=1-(1/2)k_w v_\parallel v_w^2 c^{-2}\gamma_0^2\Omega_0^{-1}, \tag{48}$$

$$\Psi=1-(1/2)\gamma_\parallel^2 v_w^2 c^{-2}, \tag{49}$$

$$\Phi = 1 - \gamma_\parallel^2 \Omega_0 \Omega_w v_w v_\parallel^{-1} [(v_\parallel v_w^{-1} \Omega_w + v_w^2 v_\parallel^{-2} \Omega_0) v_\parallel v_w^{-1} \Omega_w - (\bar{\omega}^2 + \delta_1)]^{-1} - \delta_2 [(\Omega_0 - k_w v_\parallel)^2 + \Omega_0 \Omega_w v_w v_\parallel^{-1} - (\bar{\omega}^2 + \delta_1)]^{-1}. \tag{50}$$

The quantities δ_1 and δ_2 vanish in the limit of infinite beam radius and also in the limit of the zero wiggler. They are given by a hierarchy of algebraic equations which will be omitted for brevity. Note that ω_b is the nonrelativistic beam plasma frequency in the lab frame.

Imposing the boundary condition on the surface of the beam will give an additional equation which should be solved simultaneously with Eq. (45) to obtain ω as a function of k. In a completely filled waveguide, $a/R = 1$, continuity of δE_z on the surface of the beam gives $\kappa = p_{0\nu}/R$, where $p_{0\nu}$ is the νth zero of J_0. Equation (45) with $\kappa = p_{0\nu}/R$ is the dispersion relation of the space-charge waves in a completely filled waveguide and with the wiggler present. In this case $(a/R = 1)$ and in the limit of zero wiggler, Eq. (45), when transformed to the beam frame, gives the dispersion relation for axially symmetric space charge waves [4].

IV. PARTIALLY FILLED WAVEGUIDE

A. With the wiggler present

In a waveguide which is partially filled with the electron beam $(a/R < 1)$ the lab frame electric and magnetic fields, in the vacuum region $(a < r < R)$, may be written as

$$\delta E_z^v = \delta E_{z0} J_0(\kappa a) \frac{[I_0(k_B r) K_0(k_B R) - I_0(k_B R) K_0(k_B r)]}{[I_0(k_B a) K_0(k_B R) - I_0(k_B R) K_0(k_B a)]} \times \exp[i(kz - \omega t)], \tag{51}$$

$$\delta E_r^v = -i \gamma_\parallel \delta E_{z0} J_0(\kappa a) \times \frac{[I_1(k_B r) K_0(k_B R) + I_0(k_B R) K_1(k_B r)]}{[I_0(k_B a) K_0(k_B R) - I_0(k_B R) K_0(k_B a)]} \times \exp[i(kz - \omega t)], \tag{52}$$

$$\delta B_\theta^v = v_\parallel c^{-1} \delta E_r^v. \tag{53}$$

Here the axial component of the electric field vanishes at the guide surface and is continuous on the beam surface, and $k_B = \gamma_\parallel (k - \omega v_\parallel c^{-2})$ is the beam-frame wave number. The discontinuity of radial current density on the surface of the beam, which is caused, to zeroth order, by the nonzero δv_r at $r = a$ [Eq. (25)], produces a surface charge density on the beam surface. In order to find the discontinuity of the radial component of electric field, Gauss's law in the beam frame

$$\nabla_B \cdot \delta \mathbf{E}_B = 4\pi \delta\rho_B \tag{54}$$

will be differentiated with respect to time to be written as

$$\nabla_B \cdot \left[\frac{\partial}{\partial t_B} \delta \mathbf{E}_B + 4\pi \delta \mathbf{J}_B \right] = 0, \tag{55}$$

where the indices B refer to the beam frame quantities. By integrating Eq. (55) over the volume of a small pillbox and applying the divergence theorem the following relation, at the surface of the beam, is found

$$\frac{\partial}{\partial t_B} (\delta E_{rB}^v - \delta E_{rB}) = 4\pi \delta J_{rB} \quad (r = a). \tag{56}$$

This equation will next be transformed and written in terms of the lab frame quantities, using Eqs. (28)–(38) and Eqs. (51)–(53). To zeroth order, this will yield

$$i\bar{\omega} \gamma_\parallel^2 \delta E_{r0} - \bar{\omega} S \gamma_\parallel \delta E_{z0} + 4\pi e n_0 \delta v_{r0} - i\bar{\omega} v_\parallel c^{-1} \gamma_\parallel^2 \delta B_{\theta 0} = 0, \tag{57}$$

where

$$S = -\frac{J_0(\kappa a)}{J_1(\kappa a)} \frac{[I_1(k_B a) K_0(k_B R) + I_0(k_B R) K_1(k_B a)]}{[I_0(k_B a) K_0(k_B R) - I_0(k_B R) K_0(k_B a)]}. \tag{58}$$

To solve Eq. (57) δE_{r0}, δv_{r0}, and $\delta B_{\theta 0}$ will be written in terms of δE_{z0}, with the aid of the results in Sec. III, to obtain

$$\frac{\bar{\omega}\kappa}{k - \omega v_\parallel c^{-2}} \left[1 - \frac{\omega_b^2 \Psi \gamma_0^{-1} \gamma_\parallel^{-2}}{(\bar{\omega}^2 - b^2 \Omega_0^2)} \right] - \bar{\omega} \gamma_\parallel S + \delta_3 = 0, \tag{59}$$

where δ_3, which vanishes in the limit of infinite beam radius and also in the limit of zero wiggler, is given by a hierarchy of algebraic equations and, therefore, will be omitted for brevity. The simultaneous solution of Eqs. (45) and (59) yields the dispersion relation between k and ω for the space-charge waves when their phase velocity, in the beam frame, is small compared to the speed of light.

For the completely filled guide $(a/R = 1)$ the denominator in Eq. (58) becomes zero, and $J_0(\kappa a)$ in the numerator should be zero, which gives $\kappa = p_{0\nu}/R$. In the limit of infinite beam radius, S is of the order of $J_0(\kappa a)/J_1(\kappa a)$ and all four terms in Eq. (59) vanish when $\kappa = p_{0\nu}/R$ which gives $\kappa \to 0$. This result may also be seen from Eqs. (18)–(27) which are expected to be independent of r in the limit of infinite beam radius. The only possibility is $\kappa = 0$.

B. Without the wiggler

In this problem analysis is carried out in the lab frame. Equation (45) is obtained by solving the modified form of Gauss's law, Eq. (15), in the lab frame. To find Eq. (59), however, Gauss's law Eq. (54), in the beam frame, is integrated over the beam surface and the result is then transformed to the lab frame. Equations (45) and (59) in the limit of zero wiggler reduce to

$$\frac{\kappa^2}{k_B^2} + \frac{(\Omega_{0B}^2 - \omega_B^2)(\omega_B^2 - \omega_{PB}^2)}{\omega_B^2 (\Omega_{0B}^2 + \omega_{PB}^2 - \omega_B^2)} = 0, \tag{60}$$

$$S - \frac{\kappa}{k_B} \frac{\Omega_{0B}^2 + \omega_{PB}^2 - \omega_B^2}{\Omega_{0B}^2 - \omega_B^2} = 0. \tag{61}$$

Here $\omega_B = \gamma_\parallel (\omega - k v_\parallel)$ is the wave frequency, $\Omega_{0B} = eB_0/m_0 c$ is the cyclotron frequency, and ω_{PB} is the plasma frequency, all in the beam frame. Equations (60) and (61) comprise the beam frame dispersion relation for axially symmetric, space-charge waves in a partially filled waveguide, first reported by Trivelpiece and Gould [6]. Their simultaneous solutions which are obtained numerically were found to be in close agreement with the full electromagnetic treatment [7] at large wave numbers.

It is also instructive to apply the boundary condition in the lab frame by integrating the modified form of Gauss's law over the beam surface. This gives

$$S-\frac{\kappa}{k_B}\,\frac{\Omega_{0B}^2-\omega_B^2+\omega_{PB}^2(1+k_B\omega_B^{-1}v_\parallel)^{-1}}{\Omega_{0B}^2-\omega_B^2}=0, \tag{62}$$

which contains the erroneous factor $k_B\omega_B^{-1}v_\parallel$ compared to Eq. (61). The reason is that the relativistic axial velocity of the surface-charge density, when viewed in the lab frame, produces a large surface-current density, which is neglected in deriving Eq. (62). Only when $v_\parallel$ is vanishingly small is the surface current density as well as the extra factor $k_B\omega_B^{-1}v_\parallel$, in Eq. (62), negligible and Eq. (62) reduces to Eq. (61).

V. NUMERICAL RESULTS

In order to find the dispersion relation for space-charge waves in a partially filled plasma waveguide, with the effect of the wiggler included, simultaneous solution of the two nonlinear equations (45) and (59) was attempted. The numerical solutions turned out to be unstable for a wide range of parameters. Therefore, an alternative procedure was employed. The quantities $\delta B_{\theta 0}$, δE_{r0}, and δn_0 are eliminated from Eqs. (28)–(38) using Eqs. (29)–(31). The system of equations for the remaining eight unknowns may be written in a matrix form as follows:

$$\begin{pmatrix}
-i\overline{\omega} & \frac{v_w}{2}\left(\frac{\alpha_3}{\alpha_1}-\kappa\right) & 0 & \Omega_0+\frac{\eta}{2} & 0 & -\frac{v_w}{2}\frac{\alpha_3}{\alpha_1} & 0 & \frac{i\kappa\Psi}{\gamma_0\gamma_\parallel^2\overline{k}} \\
v_w(\kappa-\alpha_2) & -i\overline{\omega} & k_wv_\parallel+\frac{\eta}{4} & 0 & \Omega_0+\frac{\eta}{4} & 0 & 0 & \frac{-v_wv_\parallel}{\gamma_0c^2} \\
0 & -k_wv_\parallel+\frac{\eta}{4} & -i\overline{\omega} & -v_w\alpha_2 & 0 & \Omega_0+\frac{3\eta}{4} & -\Gamma & 0 \\
-\Omega_0-\frac{\eta}{2} & 0 & \frac{v_w}{2}\frac{\alpha_3}{\alpha_1} & -i\overline{\omega} & \frac{v_w}{2}\left(\frac{\alpha_3}{\alpha_1}-\kappa\right) & 0 & 0 & 0 \\
0 & -\Omega_0-\frac{3\eta}{4} & 0 & v_w(\kappa-\alpha_2) & -i\overline{\omega} & k_wv_\parallel-\frac{\eta}{4} & \Gamma & 0 \\
v_w\alpha_2 & 0 & -\Omega_0-\frac{\eta}{4} & 0 & -k_wv_\parallel-\frac{\eta}{4} & -i\overline{\omega} & 0 & \frac{-v_wv_\parallel}{\gamma_0c^2} \\
0 & 0 & \Omega_w/2 & 0 & -\Omega_w/2 & 0 & -i\overline{\omega} & \frac{1}{\gamma_0\gamma_\parallel^2} \\
\frac{i\kappa}{\gamma_\parallel^2\overline{k}} & 0 & 0 & 0 & 0 & 0 & -1 & \frac{-i\overline{\omega}}{\omega_b^2}\left[1+\frac{\kappa^2}{\gamma_\parallel^2\overline{k^2}}\right]
\end{pmatrix}
\times\begin{pmatrix}
\delta v_{r0}\\ \delta v_{r1}\\ \delta v_{r2}\\ \delta v_{\theta 0}\\ \delta v_{\theta 1}\\ \delta v_{\theta 2}\\ \delta v_{z0}\\ \frac{e}{m_0}\delta E_{z0}
\end{pmatrix}=0, \tag{63}$$

with

$$\Gamma \equiv k_w v_w + \Omega_w - \eta \frac{v_\parallel}{v_w}. \tag{64}$$

For nontrivial solution the 8×8 matrix in Eq. (63), which will be denoted by C_1 and is formed by the coefficients of the field components, should have zero determinant, which yields

$$\det C_1 = 0. \tag{65}$$

An additional equation, which can be obtained from the boundary conditions, is required to account for κ in Eq. (65). The quantities $\delta B_{\theta 0}$ and δE_{r0} are eliminated from Eq. (57), using Eqs. (29) and (30), to obtain

$$\bar{\omega}\gamma_\parallel \left[S - \frac{\kappa}{\gamma_\parallel \bar{k}} \right] \frac{e}{m_0} \delta E_{z0} + \omega_b^2 \delta v_{r0} = 0. \tag{66}$$

The matrix obtained by replacing the elements of the first row of matrix C_1 with the corresponding coefficients in Eq. (66) will be denoted by C_2. The determinant of this matrix should also be set equal to zero, which gives

$$\det C_2 = 0. \tag{67}$$

Simultaneous solution of Eqs. (65) and (67) yields the dispersion relation between k and ω. One advantage in solving Eqs. (65) and (67) instead of Eqs. (45) and (59) is that their solutions are numerically more stable and the second advantage is that there is no need for δ_1, δ_2, and δ_3, in this case, which require long chains of equations. Although δ_1, δ_2, and δ_3 were not used in solving the coupled equations to obtain the dispersion relation, they will be presented elsewhere [8]. These quantities were used to calculate the density factor Φ and the radius factor ρ.

Numerical calculations have been made to illustrate the effects of the ratio of the beam radius to waveguide radius, wiggler magnetic field, and axial magnetic field on the plasmalike waves with large beam-frame wave numbers. These are space-charge waves for which the beam-frame frequency approaches the plasma frequency as $k_B R \to \infty$ in the absence of the wiggler field. Wiggler magnetic field B_w and wiggler wave length $2\pi/k_w$ were taken to be 760 G and 5 cm, respectively. The inner radius R of the waveguide was taken to be 0.3 cm. Lab-frame electron density n_0 was taken to be $10^{12}\,\mathrm{cm}^{-3}$ and electron beam energy $(\gamma_0 - 1)m_0c^2$ was taken to be 700 keV corresponding to a Lorentz factor γ_0 of 2.37. Axial magnetic field B_{0z} was varied from 0 to 25.4 kG which corresponds to a variation from 0 to 5 in the normalized lab-frame relativistic cyclotron frequency Ω_0/ck_w associated with B_{0z}. Three values for the ratio of the beam radius to the waveguide radius were chosen, with $a/R=1$ corresponding to a guide completely filled with the electron beam and $a/R=0.6, 0.3$ corresponding to a partially filled waveguide.

Figures 1 and 2 show the waveguide radius factor ρ as functions of Ω_0/ck_w, for the group I and group II orbits, respectively. In the dispersion relation, ρ multiplies R producing an effective waveguide radius ρR. A singularity in ρ is observed in Fig. 1 for the group I orbits. This singularity

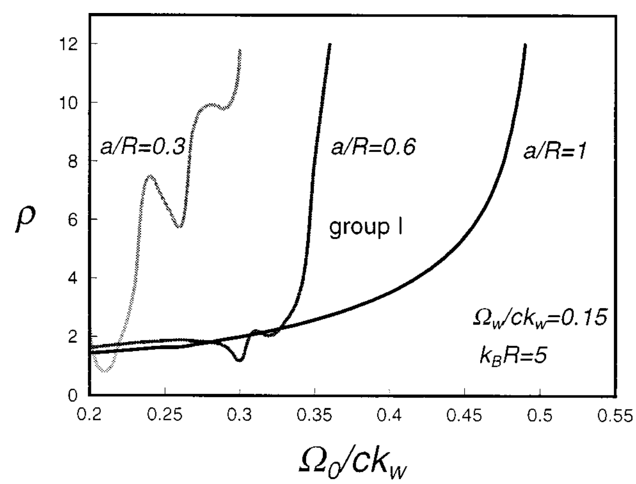

FIG. 1. Waveguide radius factor ρ as a function of the normalized axial magnetic field Ω_0/ck_w for group I orbits.

which, for the completely filled case $a/R=1$ coincides with the point of transition to the orbit instability at $\Omega_0/ck_w \cong 0.53$, occurs at $\Omega_0/ck_w \cong 0.35$ and 0.3 for the partially filled cases of $a/R=0.6$ and 0.3, respectively. No frequency was found for the cyclotron frequency Ω_0/ck_w larger than the one that makes ρ singular (see Fig. 5). No value for ρ was found in Fig. 2 for Ω_0/ck_w less than some minimum value (around 1.5 for $a/R=1$) where the frequency of the wave becomes complex; ρ which is very large at this point falls with increasing Ω_0/ck_w to small values around $\rho \cong 0.1$ and from this point ρ increases and approaches unity for large values of Ω_0/ck_w. For the large values of axial magnetic field the wiggler has no effect on the space-charge wave and $\rho=1$ is expected in this limit.

Figures 3 and 4 show the density factor Φ as a function of Ω_0/ck_w for the group I and group II orbits, respectively. No value for Φ is given, in Fig. 3, for Ω_0/ck_w larger than the values that make ρ singular in Fig. 1. Singularities for Φ, in Fig. 4, correspond to the minima for ρ in Fig. 2, which occur around $\Omega_0/ck_w \cong 1.7$, 1.9, and 2.3 for $a/R=1$, 0.6, and 0.3,

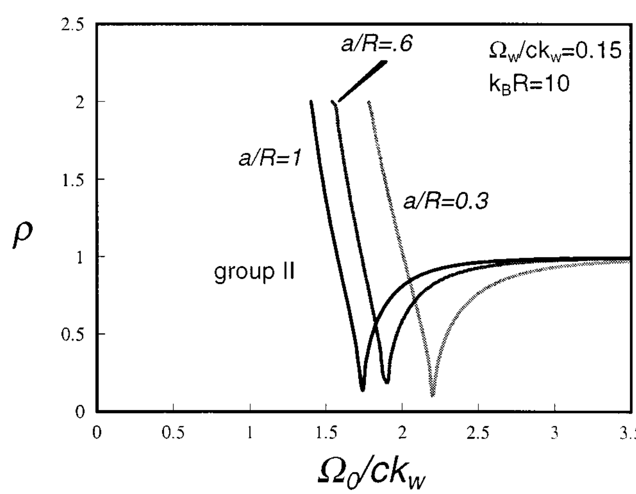

FIG. 2. Waveguide radius factor ρ as a function of the normalized axial magnetic field Ω_0/ck_w for group II orbits.

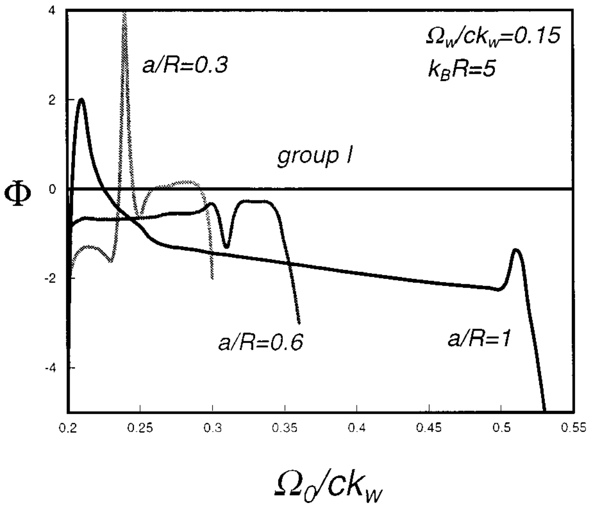

FIG. 3. Electron-density factor Φ as a function of the normalized axial magnetic field Ω_0/ck_w for group I orbits.

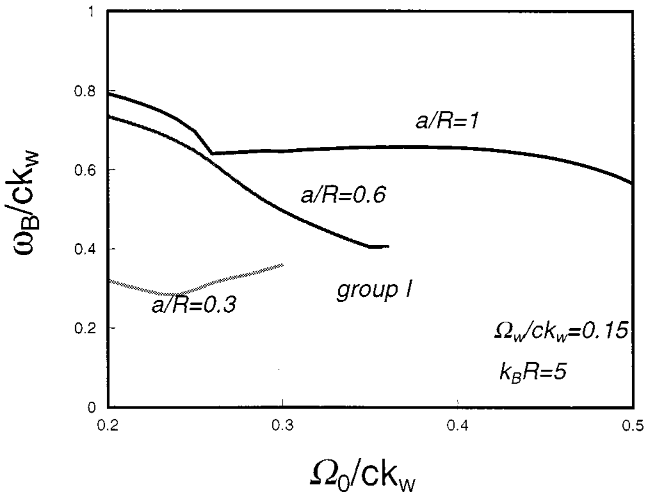

FIG. 5. Normalized beam-frame frequency ω_B/ck_w as a function of the normalized axial magnetic field Ω_0/ck_w for group I orbits.

respectively. For the large values of axial magnetic field the wiggler field has no effect on the plasmalike waves and Φ approaches unity in the infinite-magnetic-field limit.

Figures 5 and 6 illustrate the variation of the frequency of the plasmalike waves with Ω_0/ck_w for group I and group II orbits, respectively. For group I orbits in Fig. 5 real frequencies were not found for Ω_0/ck_w larger than the values that make ρ singular in Fig. 1. For group II orbits in Fig. 6 slopes of the curves change at about the values of Ω_0/ck_w that make Φ singular in Fig. 4. For Ω_0/ck_w less than some minimum value (around 1.5 for $a/R=1$) real frequencies were not found. This corresponds to the negative mass regime, at the limit of $R\rightarrow\infty$, where Φ becomes negative making the frequencies complex and the plasma waves unstable. It should be noted that although Φ is negative in the range shown in Fig. 4, real frequencies were found for this range in Fig. 6.

Figures 7 and 8 show the dispersion curves of the plasmalike waves for group I and group II orbits, respectively. For group I orbits, in Fig. 7, Ω_0/ck_w is chosen for each a/R to be less than the value that makes ρ singular. These values are $\Omega_0/ck_w=0.3$ for $a/R=1,0.6$ and $\Omega_0/ck_w=0.2$ for $a/R=0.3$. For the solid curves $\Omega_w/ck_w=0.1$ and for the dashed curve, the wiggler field is zero with $\Omega_w/ck_w=0$. There is almost no variation of the frequencies ω_B/ck_w with the normalized wave number k_BR; this is due to the small values of the cylcotron frequency that confine the frequencies of the plasmalike waves to a narrow region between the effective upper-hybrid frequency and the effective plasma frequency. The wiggler field in Fig. 8 is zero for the dashed curve and corresponds to $\Omega_w/ck_w=0.15$ for the solid curves. Comparing the two curves for $a/R=0.3$ (one with and the other without the wiggler), in Figs. 7 and 8, reveals that the wiggler field lowers the frequency of the plasmalike wave.

VI. DISCUSSION AND CONCLUSIONS

The present method of analysis is a generalization of the method of Ref. [5] to make it applicable to the case in which the electron beam only partially fills the waveguide. Both methods are based on a beam-frame electrostatic approximation which employs Gauss's law and the requirements that

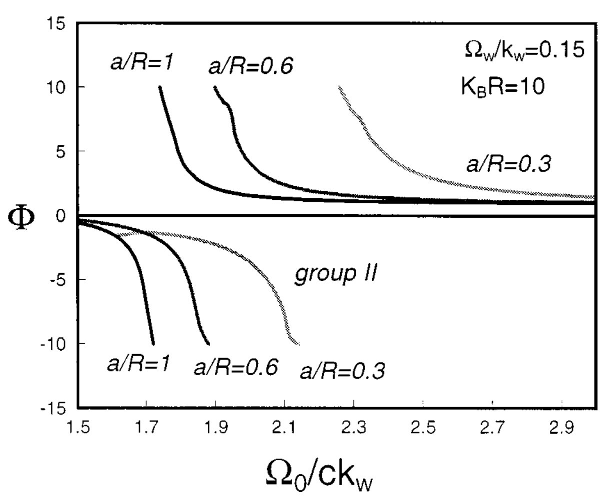

FIG. 4. Electron-density factor Φ as a function of the normalized axial magnetic field Ω_0/ck_w for group II orbits.

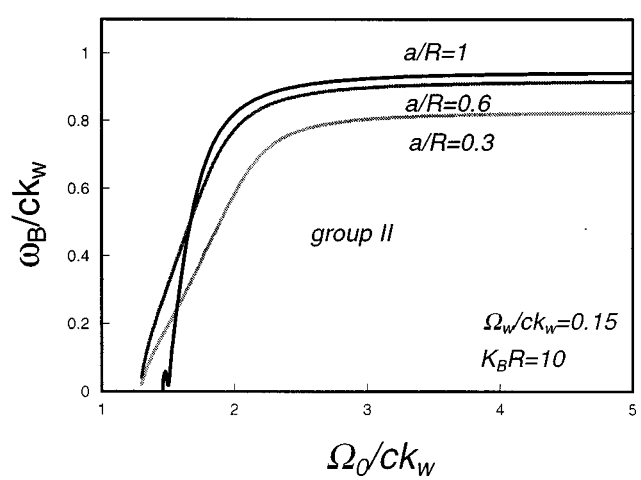

FIG. 6. Normalized beam-frame frequency ω_B/ck_w as a function of the normalized axial magnetic field Ω_0/ck_w for group II orbits.

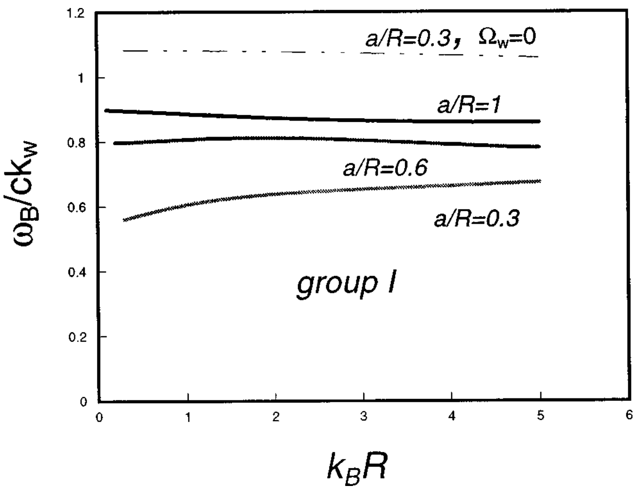

FIG. 7. Beam-frame dispersion relation for group I orbits.

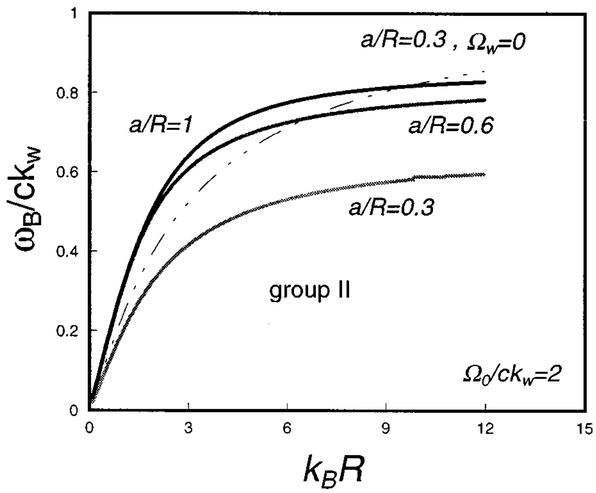

FIG. 8. Beam-frame dispersion relation for group II orbits.

the magnetic field and the curl of the electric field of the wave be zero in the beam frame. Both methods yield correct results in the zero-wiggler-field limit. In order to ensure correct results in the infinite-beam-radius limit, a modification of the form of the assumed solution for the perturbed transverse velocity components [Eqs. (25) and (26)] has been made. Consequently, the results in Ref. [5] are not identical with those of the present method when applied to a completely filled guide.

A boundary condition for the radial component of the electric field at the beam surface was derived from Gauss's law in the beam frame and then transformed into the lab frame. The corresponding boundary condition derived directly in the lab frame from the modified Gauss's law involves a surface current density. Neglecting this unknown surface current would yield a specious result which can be demonstrated by transformation into the beam frame; the boundary condition thereby obtained would contain $v_\parallel$, but $v_\parallel$ is not relevant in the beam frame.

The present analysis is based on the idealized, one-dimensional approximation in which the radial variation of the wiggler magnetic field is neglected. This is a valid approximation provided that the electron displacement from the waveguide axis is small compared to the wiggler wave length (period) which was taken herein to be 5 cm. Since the inner radius of the waveguide was taken to be 0.3 cm, the radial coordinate of each electron cannot exceed this value. Furthermore, in the calculations for the partially filled waveguide, the electron-beam radius was taken as 0.09 and 0.18 cm. Consequently, the radially uniform-wiggler approximation is excellent. The electrons will remain confined away from the waveguide wall except very near resonance (i.e., when $\Omega_0 \cong k_w v_\parallel$).

Lab-frame dispersion relation (45) has been cast into the form for space-charge waves in a plasma waveguide. To account for the effects of the wiggler field two electron-density factors Φ and Ψ, a waveguide radius factor ρ, and an axial magnetic field factor b were introduced. Only Ψ is given by the same equation as in Refs. [4] and [5]. The other three factors differ because of the different (improved) model employed herein. Note that Φ is the dominant density factor for plasmalike waves at short wavelengths (i.e., $k_B R \to \infty$). Some numerical results have been presented to illustrate the effects of the wiggler and the ratio of the beam radius to guide radius on the dispersion relation. The research reported herein is directed toward developing an accurate method of treating space-charge waves in a Raman free-electron laser. It will be used subsequently in a study of the dependence of the growth rate and radiation frequency on the system parameters.

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